# Supercoherent state representation and an inhomogeneous differential realization of the *SPL*(2,1) superalgebra



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# Supercoherent state representation and an inhomogeneous differential realization of the *SPL*(2, 1) superalgebra

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**Abstract.** The supercoherent states of the *SPL*(2, 1) superalgebra corresponding to irreducible representations of generic type are constructed, and their properties are discussed in detail. The matrix elements of the *SPL*(2, 1) generators in the supercoherent state space are calculated. An inhomogeneous differential realization of the *SPL*(2, 1) superalgebra is given.

## 1. Introduction

Coherent states of Lie (super)algebras have played an important role in the study of quantum mechanics, quantum electrodynamics, quantum optics and quantum field theory, which provide a natural link between classical and quantum phenomena and are related to the path-integral formalism [1–7]. Recently, much attention has been paid to the coherent states of Lie (super)algebras [6–10]. Quasi-exactly solvable problems (QESP) in quantum mechanics have become increasingly important because they have been generalized to study the conformal field theory [10]. A connection between QESP and finite-dimensional inhomogeneous differential realizations of Lie algebras (or superalgebras) has been described for the first time by Turbiner [11–14]. Turbiner gave a complete classification of the one-dimensional QESP by making use of the inhomogeneous differential realization of the $su(2)$ algebra, and pointed out that the multi-dimensional QESP may be studied and the general procedure to construct the multi-dimensional QESP in terms of the inhomogeneous differential realizations of the Lie superalgebras was presented [11–15]. The key to the solution of the QESP lies in studying finite-dimensional inhomogeneous differential realizations of Lie (super)algebras. Therefore, it is very important to study the inhomogeneous differential realizations of Lie superalgebras. The purpose of the present paper is to derive further new inhomogeneous differential realizations of the *SPL*(2, 1) superalgebras on the basis of studying the supercoherent state. We shall first construct the coherent states of the *SPL*(2, 1) superalgebra corresponding to irreducible representations of generic type, and discuss their properties. Then we calculate the matrix elements of the *SPL*(2, 1) generators in the supercoherent state representation and give a new form of the inhomogeneous differential realizations of the *SPL*(2, 1) in the supercoherent state space. We do not consider here the special one-parameter family of four-dimensional irreducible representations of the *SPL*(2, 1), nor the large class of indecomposable finite-dimensional representations [16].

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## 2. The $SPL(2,1)$ supercoherent state and properties

In accordance with [17], the generators of the $SPL(2, 1)$ superalgebra read as follows:

$$\{Q_3, Q_+, Q_-, B \in SPL(2,1)_{\bar{0}} | V_+, V_-, W_+, W_- \in SPL(2,1)_{\bar{1}}\} \tag{1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
&[Q_3, Q_\pm] = \pm Q_\pm \qquad [Q_+, Q_-] = 2Q_3 \qquad [B, Q_\pm] = [B, Q_3] = 0\\
&[Q_3, V_\pm] = \pm\tfrac{1}{2}V_\pm \qquad [Q_3, W_\pm] = \pm\tfrac{1}{2}W_\pm \qquad [B, V_\pm] = \tfrac{1}{2}V_\pm\\
&[B, W_\pm] = -\tfrac{1}{2}W_\pm \qquad [Q_\pm, V_\mp] = V_\pm \qquad [Q_\pm, W_\mp] = W_\pm\\
&[Q_\pm, V_\pm] = 0 \qquad [Q_\pm, W_\pm] = 0\\
&\{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0\\
&\{V_\pm, W_\pm\} = \pm Q_\pm \qquad \{V_\pm, W_\mp\} = -Q_3 \pm B.
\end{aligned} \tag{2}$$

According to [18] and relabelling the basis vector $\varphi(k, \alpha_1, \alpha_2)$ of the finite-dimensional irreducible representation of the $SPL(2, 1)$ superalgebra by $|N, k, \alpha_1, \alpha_2\rangle$ the actions of the generators on the basis vectors are

$$\begin{aligned}
Q_3|N,k,\alpha_1,\alpha_2\rangle &= \left(-\frac{N}{2} + k + \frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right)|N,k,\alpha_1,\alpha_2\rangle\\
B|N,k,\alpha_1,\alpha_2\rangle &= \tfrac{1}{2}(\alpha_2 - \alpha_1)|N,k,\alpha_1,\alpha_2\rangle\\
Q_+|N,k,\alpha_1,\alpha_2\rangle &= (N - k - \alpha_1 - \alpha_2)|N,k+1,\alpha_1,\alpha_2\rangle\\
Q_-|N,k,\alpha_1,\alpha_2\rangle &= k|N,k-1,\alpha_1,\alpha_2\rangle\\
V_+|N,k,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(N-k-\alpha_1)(1-\alpha_2)|N,k,\alpha_1,\alpha_2+1\rangle\\
&\quad + \frac{1}{\sqrt{2}}\alpha_1|N,k+1,\alpha_1-1,\alpha_2\rangle\\
V_-|N,k,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}\alpha_1|N,k,\alpha_1-1,\alpha_2\rangle - \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(1-\alpha_2)k|N,k-1,\alpha_1,\alpha_2+1\rangle\\
W_+|N,k,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}(N-k-\alpha_2)(1-\alpha_1)|N,k,\alpha_1+1,\alpha_2\rangle\\
&\quad + \frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2|N,k+1,\alpha_1,\alpha_2-1\rangle\\
W_-|N,k,\alpha_1,\alpha_2\rangle &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2|N,k,\alpha_1,\alpha_2-1\rangle - \frac{1}{\sqrt{2}}(1-\alpha_1)k|N,k-1,\alpha_1+1,\alpha_2\rangle
\end{aligned} \tag{3}$$

where

$$\{|N,k,\alpha_1,\alpha_2\rangle | k + \alpha_1 + \alpha_2 \leqslant N,\ N \in Z^+,\ k = 0,1,2,\ldots,\alpha_1,\alpha_2 = 0,1\}$$

and

$$k = \begin{cases} 0,1,\ldots,N & \text{when}\quad \alpha_1 = 0 \quad \alpha_2 = 0\\ 0,1,\ldots,N-1 & \text{when}\quad \alpha_1 = 0 \quad \alpha_2 = 1\\ 0,1,\ldots,N-1 & \text{when}\quad \alpha_1 = 1 \quad \alpha_2 = 0\\ 0,1,\ldots,N-2 & \text{when}\quad \alpha_1 = 1 \quad \alpha_2 = 1. \end{cases} \tag{4}$$

The space $\{|N, k, \alpha_1, \alpha_2\rangle\}$ of the irrep $N$ of the $SPL(2, 1)$ superalgebra is $4N$ dimensional and may be divided into four subspaces $\{|N, k, 0, 0\rangle\}$, $\{|N, k, 0, 1\rangle\}$, $\{|N, k, 1, 0\rangle\}$ and $\{|N, k, 1, 1\rangle\}$ corresponding to $(\alpha_1, \alpha_2) = (0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$, respectively. All the basis vectors $|N, k, \alpha_1, \alpha_2\rangle$ are assumed to be normalized as

$$\begin{aligned} &\binom{N}{k}\langle N, k, 0, 0|N, k, 0, 0\rangle = 1 && \binom{N-1}{k}\langle N, k, 0, 1|N, k, 0, 1\rangle = 1 \\ &\binom{N-1}{k}\langle N, k, 1, 0|N, k, 1, 0\rangle = 1 && \binom{N-2}{k}\langle N, k, 1, 1|N, k, 1, 1\rangle = 1. \end{aligned} \tag{5}$$

The completeness condition of the vectors of the irrep may be expressed as

$$\begin{aligned} &\sum_{k=0}^{N}\binom{N}{k}|N, k, 0, 0\rangle\langle N, k, 0, 0| + \sum_{k=0}^{N-1}\binom{N-1}{k}|N, k, 0, 1\rangle\langle N, k, 0, 1| \\ &\qquad + \sum_{k=0}^{N-1}\binom{N-1}{k}|N, k, 1, 0\rangle\langle N, k, 1, 0| \\ &\qquad + \sum_{k=0}^{N-2}\binom{N-2}{k}|N, k, 1, 1\rangle\langle N, k, 1, 1| = I \end{aligned} \tag{6}$$

where $I$ is the identity operator.

One can easily show the following formulae from equation (3):

$$\begin{aligned} Q_+^n|N, 0, 0, 0\rangle &= \binom{N}{n} n!|N, n, 0, 0\rangle \\ Q_+^n|N, 0, 0, 1\rangle &= \binom{N-1}{n} n!|N, n, 0, 1\rangle \\ Q_+^n|N, 0, 1, 0\rangle &= \binom{N-1}{n} n!|N, n, 1, 0\rangle \\ Q_+^n|N, 0, 1, 1\rangle &= \binom{N-2}{n} n!|N, n, 1, 1\rangle \end{aligned} \tag{7}$$

where

$$\binom{N}{n} = \frac{N!}{(N-n)!n!}.$$

In terms of Bloch's method we now define the supercoherent state $|Z, \xi_1, \xi_2\rangle$ by applying the exponential operator $\exp(ZQ_+ + \xi_1 V_+ + \xi_2 W_+)$ on the lowest-weight state $|N, 0, 0, 0\rangle$ of the $SPL(2, 1)$ irrep

$$|Z, \xi_1, \xi_2\rangle = H(z, \xi_1, \xi_2)\exp(ZQ_+ + \xi_1 V_+ + \xi_2 W_+)|N, 0, 0, 0\rangle \tag{8}$$

where $H(z, \xi_1, \xi_2)$ is a normalization factor and its modulus squared is assumed to include only even-order terms for $\xi_1$ and $\xi_2$, and $Z$, $\xi_1$ and $\xi_2$ are one complex variable and two Grassmann variables, respectively. Considering the generator $Q_+$ as commutable with $V_+$ and $W_+$, and the anticommutation relation of the Grassmann variables $\xi_1$, $\xi_2$,

$$\{\xi_1, \xi_2\} = 0 \tag{9}$$

we can easily show the following formula:

$$\exp(ZQ_+ + \xi_1 V_+ + \xi_2 W_+) = \exp((Z - \tfrac{1}{2}\xi_1\xi_2)Q_+)\exp(\xi_1 V_+)\exp(\xi_2 W_+). \tag{10}$$

Using the formulae (7) and (10), the supercoherent state equation (8) may be rewritten as follows:

$$\begin{aligned}|Z,\xi_1,\xi_2\rangle &= H(z,\xi_1,\xi_2)\Bigg\{\sum_{n=0}^{N}\binom{N}{n}Z^n|N,n,0,0\rangle+\frac{1}{\sqrt{2}}N\xi_1\sum_{n=0}^{N-1}\binom{N-1}{n}Z^n|N,n,0,1\rangle\\ &\quad+\frac{1}{\sqrt{2}}N\xi_2\sum_{n=0}^{N-1}\binom{N-1}{n}Z^n|N,n,1,0\rangle\\ &\quad-\frac{1}{2}N(N-1)\xi_1\xi_2\sum_{n=0}^{N-2}\binom{N-2}{n}Z^n|N,n,1,1\rangle\Bigg\}\\ &= H(z,\xi_1,\xi_2)\Bigg\{|Z\rangle_1+\frac{1}{\sqrt{2}}N\xi_1|Z\rangle_2+\frac{1}{\sqrt{2}}N\xi_2|Z\rangle_3-\frac{1}{2}N(N-1)\xi_1\xi_2|Z\rangle_4\Bigg\} \qquad (11)\end{aligned}$$

where $|Z\rangle_1$, $|Z\rangle_2$, $|Z\rangle_3$ and $|Z\rangle_4$ are four simple coherent states associated with four subspaces $\{|N,k,0,0\rangle\}$, $\{|N,k,0,1\rangle\}$, $\{|N,k,1,0\rangle\}$ and $\{|N,k,1,1\rangle\}$ of the $SPL(2,1)$ irrep,

$$\begin{aligned}|Z\rangle_1 &= \sum_{n=0}^{N}\binom{N}{n}Z^n|N,n,0,0\rangle \qquad & |Z\rangle_2 &= \sum_{n=0}^{N-1}\binom{N-1}{n}Z^n|N,n,0,1\rangle\\ |Z\rangle_3 &= \sum_{n=0}^{N-1}\binom{N-1}{n}Z^n|N,n,1,0\rangle \qquad & |Z\rangle_4 &= \sum_{n=0}^{N-2}\binom{N-2}{n}Z^n|N,n,1,1\rangle.\end{aligned} \qquad (12)$$

According to equation (11) we have

$$\langle Z,\xi_1,\xi_2| = \Bigg\{{}_1\langle Z|+\frac{1}{\sqrt{2}}N\,{}_2\langle Z|\bar{\xi}_1+\frac{1}{\sqrt{2}}N\,{}_3\langle Z|\bar{\xi}_2-\frac{1}{2}N(N-1)\,{}_4\langle Z|\bar{\xi}_2\bar{\xi}_1\Bigg\}H(z,\xi_1,\xi_2) \qquad (13)$$

where $\bar{\xi}_1$, $\bar{\xi}_2$ are the complex conjugation of $\xi_1$, $\xi_2$ and we may write the scalar product of two such states as ${}_i\langle Z'|Z\rangle_i$. We see from equation (12) that these scalar products are

$$\begin{aligned}&{}_1\langle Z'|Z\rangle_1=(1+\bar{Z}'Z)^N \qquad {}_2\langle Z'|Z\rangle_2=(1+\bar{Z}'Z)^{N-1}\\ &{}_3\langle Z'|Z\rangle_3=(1+\bar{Z}'Z)^{N-1} \qquad {}_4\langle Z'|Z\rangle_4=(1+\bar{Z}'Z)^{N-2}\\ &{}_i\langle Z'|Z\rangle_j=0 \qquad (i\neq j, i, j=1,2,3,4)\end{aligned} \qquad (14)$$

which means that the two simple coherent states with different $Z$ in the same subspace are not orthogonal to each other. Nevertheless, two coherent states in different subspaces are orthogonal to each other.

Similarly, the scalar product of the supercoherent state is written as follows:

$$\begin{aligned}\langle Z',\xi_1',\xi_2'|Z,\xi_1,\xi_2\rangle &= H(z',\xi_1',\xi_2')H(z,\xi_1,\xi_2)\big\{[1+\bar{Z}'Z+\tfrac{1}{2}N^2(\bar{\xi}_1'\xi_1+\bar{\xi}_2'\xi_2)](1+\bar{Z}'Z)\\ &\quad+\tfrac{1}{4}N^2(N-1)^2\bar{\xi}_1'\xi_1\bar{\xi}_2'\xi_2\big\}(1+\bar{Z}'Z)^{N-2}. \qquad (15)\end{aligned}$$

Making $Z'=Z$, $\xi_1'=\xi_1$, $\xi_2'=\xi_2$ in equation (15) we may write the orthogonality relation of the supercoherent state $|Z,\xi_1,\xi_2\rangle$,

$$\begin{aligned}\langle Z,\xi_1,\xi_2|Z,\xi_1,\xi_2\rangle &= H(z,\xi_1,\xi_2)H(z,\xi_1,\xi_2)\big\{[1+\bar{Z}Z+\tfrac{1}{2}N^2(\bar{\xi}_1\xi_1+\bar{\xi}_2\xi_2)](1+\bar{Z}Z)\\ &\quad+\tfrac{1}{4}N^2(N-1)^2\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2\big\}(1+\bar{Z}Z)^{N-2}. \qquad (16)\end{aligned}$$

If the supercoherent state is normalized so that $\langle Z,\xi_1,\xi_2|Z,\xi_1,\xi_2\rangle=1$ we may evidently define its normalization factor by choosing

$$\begin{aligned}H(z,\xi_1,\xi_2) &= \big\{[(1+\bar{Z}Z)^N+\tfrac{1}{2}N^2(\bar{\xi}_1\xi_1+\bar{\xi}_2\xi_2)](1+\bar{Z}Z)^{N-1}\\ &\quad+\tfrac{1}{4}N^2(N-1)^2\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2(1+\bar{Z}Z)^{N-2}\big\}^{-1/2}. \qquad (17)\end{aligned}$$

The expansion coefficients of the supercoherent state $|Z, \xi_1, \xi_2\rangle$ may be found in terms of the complete orthonormal set $\{|N, k, \alpha_1, \alpha_2\rangle\}$. Thus, we have

$$\begin{aligned}
\langle Z, \xi_1, \xi_2|N, k, 0, 0\rangle &= \bar{Z}^k H(z, \xi_1, \xi_2)\\
\langle Z, \xi_1, \xi_2|N, k, 0, 1\rangle &= \frac{1}{\sqrt{2}} N\bar{\xi}_1 \bar{Z}^k H(z, \xi_1, \xi_2)\\
\langle Z, \xi_1, \xi_2|N, k, 1, 0\rangle &= \frac{1}{\sqrt{2}} N\bar{\xi}_2 \bar{Z}^k H(z, \xi_1, \xi_2)\\
\langle Z, \xi_1, \xi_2|N, k, 1, 1\rangle &= \tfrac{1}{2} N(N-1)\bar{\xi}_1\bar{\xi}_2 \bar{Z}^k H(z, \xi_1, \xi_2).
\end{aligned} \tag{18}$$

While orthogonality is a convenient property for a set of basis vectors it is not a necessary one. The essential property of such a set is that it should be complete. Since the $4N$ state vectors $\{|N, k, \alpha_1, \alpha_2\rangle\}$ of an irrep of the $SPL(2, 1)$ superalgebra are known to form a completeness orthogonal set, the supercoherent state $|Z, \xi_1, \xi_2\rangle$ for the $SPL(2, 1)$ superalgebra can be shown without difficulty to form a complete set. To give a proof we need only demonstrate that the unit operator may be expressed as a suitable sum or an integral, over the superplane, of projection operators of the form $|Z, \xi_1, \xi_2\rangle\langle Z, \xi_1, \xi_2|$. In order to describe such an integral we introduce generally the differential element of weight area in the superplane

$$d^2Z\, d^2\xi_1\, d^2\xi_2\, \sigma(Z, \xi_1, \xi_2) = |Z|\, d|Z|\, d\theta\, d\bar{\xi}_1\, d\xi_1\, d\bar{\xi}_2\, d\xi_2\, \sigma(Z, \xi_1, \xi_2) \tag{19}$$

where $\sigma(Z, \xi_1, \xi_2)$ is a weight superfield function, and $Z = |Z|e^{i\theta}$.

The problem here may by changed to that of finding the weight superfield function $\sigma(Z, \xi_1, \xi_2)$ such that

$$\begin{aligned}
&\int d^2Z\, d^2\xi_1\, d^2\xi_2\, \sigma(Z, \xi_1, \xi_2)|Z, \xi_1, \xi_2\rangle\langle Z, \xi_1, \xi_2|\\
&\quad = \sum_{k=0}^{N} \binom{N}{k} |N, k, 0, 0\rangle\langle N, k, 0, 0| + \sum_{k=0}^{N-1} \binom{N-1}{k} |N, k, 0, 1\rangle\langle N, k, 0, 1|\\
&\quad\quad + \sum_{k=0}^{N-1} \binom{N-1}{k} |N, k, 1, 0\rangle\langle N, k, 1, 0|\\
&\quad\quad + \sum_{k=0}^{N-2} \binom{N-2}{k} |N, k, 1, 1\rangle\langle N, k, 1, 1| = 1
\end{aligned} \tag{20}$$

where

$$d^2Z = |Z|\, d|Z|\, d\theta \qquad d^2\xi_1 = d\bar{\xi}_1\, d\xi_1 \qquad d^2\xi_2 = d\bar{\xi}_2\, d\xi_2.$$

To determine $\sigma(Z, \xi_1, \xi_2)$ we expand $\sigma(Z, \xi_1, \xi_2)$ in $\xi_1, \xi_2$, and save four effective items for the integral equation (20), i.e.

$$\sigma(Z, \xi_1, \xi_2) = H(z, \xi_1, \xi_2)^{-2}\{A(Z) + B(Z)\bar{\xi}_1\xi_1 + C(Z)\bar{\xi}_2\xi_2 + D(Z)\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2 \tag{21}$$

where $A(Z)$, $B(Z)$, $C(Z)$ and $D(Z)$ are four expansion coefficients. Substituting the definition of the simple coherent state equation (12) into equation (20) and integrating over the entire area of the superplane we have

$$\begin{aligned}
&\int d^2Z\, d^2\xi_1\, d^2\xi_2\, \sigma(Z, \xi_1, \xi_2)|Z, \xi_1, \xi_2\rangle\langle Z, \xi_1, \xi_2|\\
&\quad = \int d^2Z\, D(Z)|Z\rangle_{11}\langle Z| + \tfrac{1}{2}N^2 \int d^2Z\, C(Z)|Z\rangle_{22}\langle Z|
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{2}N^2\int \mathrm{d}^2Z\, B(Z)|Z\rangle_{33}\langle Z| + \tfrac{1}{4}N^2(N-1)^2\int \mathrm{d}^2Z\, A(Z)|Z\rangle_{44}\langle Z| \\
&= 2\pi\sum_{n=0}^{N}\binom{N}{n}\binom{N}{n}\int_0^\infty D(Z)|Z|^{2n+1}\,\mathrm{d}|Z|\,|N,n,0,0\rangle\langle N,n,0,0| \\
&+2\pi\sum_{n=0}^{N-1}\tfrac{1}{2}N^2\binom{N-1}{n}\binom{N-1}{n} \\
&\times\int_0^\infty C(Z)|Z|^{2n+1}\,\mathrm{d}|Z|\,|N,n,0,1\rangle\langle N,n,0,1| \\
&+2\pi\sum_{n=0}^{N-1}\tfrac{1}{2}N^2\binom{N-1}{n}\binom{N-1}{n} \\
&\times\int_0^\infty B(Z)|Z|^{2n+1}\,\mathrm{d}|Z|\,|N,n,1,0\rangle\langle N,n,1,0| \\
&+2\pi\sum_{n=0}^{N-2}\tfrac{1}{4}N^2(N-1)^2\binom{N-2}{n}\binom{N-2}{n} \\
&\times\int_0^\infty A(Z)|Z|^{2n+1}\,\mathrm{d}|Z|\,|N,n,1,1\rangle\langle N,n,1,1| = 1.
\end{aligned} \tag{22}
$$

In calculating the integral equation (22) we have used the following Grassmann integral:

$$
\begin{aligned}
&\int \mathrm{d}\xi_1 = \int \mathrm{d}\bar{\xi}_1 = \int \mathrm{d}\xi_2 = \int \mathrm{d}\bar{\xi}_2 = 0 \\
&\int \xi_1\,\mathrm{d}\xi_1 = \int \bar{\xi}_1\,\mathrm{d}\bar{\xi}_1 = \int \xi_2\,\mathrm{d}\xi_2 = \int \bar{\xi}_2\,\mathrm{d}\bar{\xi}_2 = 1.
\end{aligned} \tag{23}
$$

Comparing equation (22) with equation (6) we must have

$$
\begin{aligned}
&2\pi\binom{N}{n}\int_0^\infty D(Z)|Z|^{2n+1}\,\mathrm{d}|Z| = 1 \\
&\pi N^2\binom{N-1}{n}\int_0^\infty C(Z)|Z|^{2n+1}\,\mathrm{d}|Z| = 1 \\
&\pi N^2\binom{N-1}{n}\int_0^\infty B(Z)|Z|^{2n+1}\,\mathrm{d}|Z| = 1 \\
&\frac{1}{2}N^2(N-1)^2\binom{N-2}{n}\int_0^\infty A(Z)|Z|^{2n+1}\,\mathrm{d}|Z| = 1.
\end{aligned} \tag{24}
$$

With the aid of the following integral identity:

$$
\int_0^\infty \frac{x^{2n+1}}{(1+x^2)^m}\,\mathrm{d}x = \frac{n!(m-n-2)!}{2(m-1)!} \tag{25}
$$

and by comparing equation (24) with equation (25) we obtain the following expansion coefficients:

$$
\begin{aligned}
&D(Z) = \frac{N+1}{\pi(1+\bar{Z}Z)^{N+2}} \qquad C(Z) = \frac{2}{\pi N(1+\bar{Z}Z)^{N+1}} \\
&B(Z) = \frac{2}{\pi N(1+\bar{Z}Z)^{N+1}} \qquad A(Z) = \frac{4}{\pi N^2(N-1)(1+\bar{Z}Z)^{N}}.
\end{aligned} \tag{26}
$$

Substituting the above expansion coefficients into equation (21), we finally obtain the weight superfield function

$$\sigma(Z, \xi_1, \xi_2) = \frac{1}{\pi}\left[\frac{4}{N^2(N-1)} + \frac{2(2N-1)}{N(N-1)}(\bar{\xi}_1\xi_1 + \bar{\xi}_2\xi_2)(1+\bar{Z}Z)^{-1} + 4N\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2(1+\bar{Z}Z)^{-2}\right]. \tag{27}$$

We have thus shown

$$\frac{1}{\pi}\int d^2Z\, d^2\xi_1\, d^2\xi_2 \left[\frac{4}{N^2(N-1)} + \frac{2(2N-1)}{N(N-1)}\big(\bar{\xi}_1\xi_1 + \bar{\xi}_2\xi_2\big)(1+\bar{Z}Z)^{-1} + 4N\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2(1+\bar{Z}Z)^{-2}\right] |Z, \xi_1, \xi_2\rangle\, \langle Z, \xi_1, \xi_2| = 1 \tag{28}$$

which is a completeness relation for the supercoherent state of the $SPL(2,1)$ superalgebra of precisely the type desired. As a result of the above completeness relation, an arbitrary vector $|\Psi\rangle$ can be expanded in terms of the supercoherent state for the $SPL(2,1)$ superalgebra. To secure the expansion of $|\Psi\rangle$ in terms of the supercoherent state $|Z, \xi_1, \xi_2\rangle$, we multiply $|\Psi\rangle$ by the representation equation (28) of the unit operator. We then find

$$|\psi\rangle = \frac{1}{\pi}\int d^2Z\, d^2\xi_1\, d^2\xi_2 \left[\frac{4}{N^2(N-1)} + \frac{2(2N-1)}{N(N-1)}(\bar{\xi}_1\xi_1 + \bar{\xi}_2\xi_2)(1+\bar{Z}Z)^{-1} + 4N\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2(1+\bar{Z}Z)^{-2}\right] |Z, \xi_1, \xi_2\rangle\langle Z, \xi_1, \xi_2|\psi\rangle. \tag{29}$$

## 3. Matrix elements of the *SPL*(2, 1) generators

The present section will be devoted to calculating the matrix elements of the $SPL(2,1)$ generators in the supercoherent state representation. The calculation results are as follows:

$$\begin{aligned}
\langle Z', \xi_1', \xi_2'|\, Q_3\, |Z, \xi_1, \xi_2\rangle &= -\big[\tfrac{1}{2}N(1+\bar{Z}'Z)^2 + \tfrac{1}{4}N^2(N-1)(1+\bar{Z}'Z)(\bar{\xi}_1'\xi_1 + \bar{\xi}_2'\xi_2) \\
&\quad + \tfrac{1}{8}N^2(N-1)^2(N-2)\bar{\xi}_1'\xi_1\bar{\xi}_2'\xi_2\big] \\
&\quad \times H(z', \xi_1', \xi_2')H(z, \xi_1, \xi_2)(1-\bar{Z}'Z)(1+\bar{Z}'Z)^{N-3} \\
\langle Z', \xi_1', \xi_2'|\, B\, |Z, \xi_1, \xi_2\rangle &= \tfrac{1}{4}N^2(\bar{\xi}_1'\xi_1 - \bar{\xi}_2'\xi_2)H(z', \xi_1', \xi_2')H(z, \xi_1, \xi_2)(1+\bar{Z}'Z)^{N-1} \qquad (30) \\
\langle Z', \xi_1', \xi_2'|\, Q_+\, |Z, \xi_1, \xi_2\rangle &= \big[N(1+\bar{Z}'Z)^2 + \tfrac{1}{2}N^2(N-1)(1+\bar{Z}'Z)(\bar{\xi}_1'\xi_1 + \bar{\xi}_2'\xi_2) \\
&\quad + \tfrac{1}{4}N^2(N-1)^2(N-2)\bar{\xi}_1'\xi_1\bar{\xi}_2'\xi_2\big]H(z', \xi_1', \xi_2')H(z, \xi_1, \xi_2)\bar{Z}'(1+\bar{Z}'Z)^{N-3} \\
\langle Z', \xi_1', \xi_2'|\, Q_-\, |Z, \xi_1, \xi_2\rangle &= \big[N(1+\bar{Z}'Z)^2 + \tfrac{1}{2}N^2(N-1)(1+\bar{Z}'Z)(\bar{\xi}_1'\xi_1 + \bar{\xi}_2'\xi_2) \\
&\quad + \tfrac{1}{4}N^2(N-1)^2(N-2)\bar{\xi}_1'\xi_1\bar{\xi}_2'\xi_2\big]H(z', \xi_1', \xi_2')H(z, \xi_1, \xi_2)Z(1+\bar{Z}'Z)^{N-3} \\
\langle Z', \xi_1', \xi_2'|\, V_+\, |Z, \xi_1, \xi_2\rangle & \\
&= \big[\tfrac{1}{2}N(\bar{Z}'\xi_2 + N\bar{\xi}_1')(1+\bar{Z}'Z) - \tfrac{1}{4}N^2(N-1)(\bar{\xi}_1'\xi_1\xi_2\bar{Z}' - (N-1)\bar{\xi}_2'\bar{\xi}_1'\xi_2)\big] \\
&\quad \times H(z', \xi_1', \xi_2')H(z, \xi_1, \xi_2)(1+\bar{Z}'Z)^{N-2} \\
\langle Z', \xi_1', \xi_2'|\, V_-\, |Z, \xi_1, \xi_2\rangle & \\
&= \big[\tfrac{1}{2}N(\xi_2 - NZ\bar{\xi}_1')(1+\bar{Z}'Z) - \tfrac{1}{4}N^2(N-1)(\bar{\xi}_1'\xi_1\xi_2 + (N-1)\bar{\xi}_2'\bar{\xi}_1'\xi_2 Z)\big]
\end{aligned}$$

$$\times H(z', \xi_1', \xi_2') H(z, \xi_1, \xi_2)(1 + \bar{Z}'Z)^{N-2}$$

$$\langle Z', \xi_1', \xi_2' | \, W_+ \, |Z, \xi_1, \xi_2\rangle$$
$$= \left[\tfrac{1}{2}N(\bar{Z}_1'\xi_1 + N\bar{\xi}_2')(1 + \bar{Z}'Z) - \tfrac{1}{4}N^2(N-1)(\bar{\xi}_2'\xi_1\xi_2\bar{Z}' - (N-1)\bar{\xi}_2'\bar{\xi}_1'\xi_1)\right]$$
$$\times H(z', \xi_1', \xi_2') H(z, \xi_1, \xi_2)(1 + \bar{Z}'Z)^{N-2}$$

$$\langle Z', \xi_1', \xi_2' | \, W_- \, |Z, \xi_1, \xi_2\rangle$$
$$= \left[\tfrac{1}{2}N(\xi_1 - NZ\bar{\xi}_2')(1 + \bar{Z}'Z) - \tfrac{1}{4}N^2(N-1)(\bar{\xi}_2'\xi_1\xi_2 + (N-1)\bar{\xi}_2'\bar{\xi}_1'\xi_1 Z)\right]$$
$$\times H(z', \xi_1', \xi_2') H(z, \xi_1, \xi_2)(1 + \bar{Z}'Z)^{N-2}.$$

In evaluating the matrix elements, one needs to use equations (3), (5) and (12); for example,

$$\langle Z', \xi_1', \xi_2' | \, Q_3 \, |Z, \xi_1, \xi_2\rangle = \langle Z', \xi_1', \xi_2' | \, Q_3 \left[ \sum_{n=0}^{N} \binom{N}{n} |N, n, 0, 0\rangle\langle N, n, 0, 0| \right.$$
$$+ \sum_{n=0}^{N-1} \binom{N-1}{n} |N, n, 0, 1\rangle\langle N, n, 0, 1|$$
$$+ \sum_{n=0}^{N-1} \binom{N-1}{n} |N, n, 1, 0\rangle\langle N, n, 1, 0|$$
$$\left. + \sum_{n=0}^{N-2} \binom{N-2}{n} |N, n, 1, 1\rangle\langle N, n, 1, 1| \right] |Z, \xi_1, \xi_2\rangle$$
$$= \sum_{n=0}^{N} (-\tfrac{1}{2}N + n) \binom{N}{n} \langle Z', \xi_1', \xi_2' | N, n, 0, 0\rangle\langle N, n, 0, 0 | Z, \xi_1, \xi_2\rangle$$
$$+ \sum_{n=0}^{N-1} (-\tfrac{1}{2}(N-1) + n) \binom{N-1}{n} \langle Z', \xi_1', \xi_2' | N, n, 0, 1\rangle\langle N, n, 0, 1 | Z, \xi_1, \xi_2\rangle$$
$$+ \sum_{n=0}^{N-1} (-\tfrac{1}{2}(N-1) + n) \binom{N-1}{n} \langle Z', \xi_1', \xi_2' | N, n, 1, 0\rangle\langle N, n, 1, 0 | Z, \xi_1, \xi_2\rangle$$
$$+ \sum_{n=0}^{N-2} (-\tfrac{1}{2}(N-2) + n) \binom{N-2}{n} \langle Z', \xi_1', \xi_2' | N, n, 1, 1\rangle\langle N, n, 1, 1 | Z, \xi_1, \xi_2\rangle$$
$$= \sum_{n=0}^{N} (-\tfrac{1}{2}N + n) \binom{N}{n} (\bar{Z}'Z)^n H(z', \xi_1', \xi_2') H(z, \xi_1, \xi_2)$$
$$+ \sum_{n=0}^{N-1} (-\tfrac{1}{2}(N-1) + n) \binom{N-1}{n} \tfrac{1}{2}N^2\bar{\xi}_1'\xi_1(\bar{Z}'Z)^n H(z, \xi_1', \xi_2') H(z, \xi_1, \xi_2)$$
$$+ \sum_{n=0}^{N-1} (-\tfrac{1}{2}(N-1) + n) \binom{N-1}{n} \tfrac{1}{2}N^2\bar{\xi}_2'\xi_2(\bar{Z}'Z)^n H(z', \xi_1', \xi_2') H(z, \xi_1, \xi_2)$$
$$+ \sum_{n=0}^{N-2} (-\tfrac{1}{2}(N-2) + n) \binom{N-2}{n} \tfrac{1}{4}N^2(N-1)^2\bar{\xi}_1'\xi_1\bar{\xi}_2'\xi_2(\bar{Z}'Z)^n$$

$$\times H(z', \xi_1', \xi_2') H(z, \xi_1, \xi_2)$$
$$= -\big[\tfrac{1}{2}N(1+\bar{Z}'Z)^2 + \tfrac{1}{4}N^2(N-1)(1+\bar{Z}'Z)(\bar{\xi}_1'\xi_1 + \bar{\xi}_2'\xi_2)$$
$$+\tfrac{1}{8}N^2(N-1)^2(N-2)\bar{\xi}_1'\xi_1\bar{\xi}_2'\xi_2\big] H(z', \xi_1', \xi_2')$$
$$\times H(z, \xi_1, \xi_2)(1-\bar{Z}'Z)(1+\bar{Z}'Z)^{N-3}. \tag{31}$$

## 4. An inhomogeneous differential realization of *SPL*(2, 1)

In the section, we study the inhomogeneous differential realization of $SPL(2,1)$ in the supercoherent state space. For simplicity, we consider the realization of the superalgebra in the unnormalized new supercoherent state space $\{\|Z, \xi_1, \xi_2\rangle$ defined by

$$\|Z, \xi_1, \xi_2\rangle = |Z\rangle_1 + \frac{1}{\sqrt{2}}N\xi_1|Z\rangle_2 + \frac{1}{\sqrt{2}}N\xi_2|Z\rangle_3 - \frac{1}{2}N(N-1)\xi_1\xi_2|Z\rangle_4. \tag{32}$$

We now consider the actions of the $SPL(2,1)$ generators on the supercoherent state $\|Z, \xi_1, \xi_2\rangle$, i.e.

$$G\|Z, \xi_1, \xi_2\rangle = D(G)\|Z, \xi_1, \xi_2\rangle \tag{33}$$

where $G$ denotes the $SPL(2,1)$ generators and $D(G)$ the realizations of the generators.

To find explicit forms for the generator realizations, we begin by considering the relations between the simple coherent states and the supercoherent state. Using the definition of the supercoherent state equation (32), we find the following relations:

$$\begin{aligned}
|Z\rangle_1 &= \left(1 - \xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2} + \xi_1\xi_2\frac{\partial^2}{\partial\xi_2\partial\xi_1}\right)\|Z, \xi_1, \xi_2\rangle \\
|Z\rangle_2 &= \frac{\sqrt{2}}{N}\left(\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial^2}{\partial\xi_2\partial\xi_1}\right)\|Z, \xi_1, \xi_2\rangle \\
|Z\rangle_3 &= \frac{\sqrt{2}}{N}\left(\frac{\partial}{\partial\xi_2} - \xi_1\frac{\partial^2}{\partial\xi_1\partial\xi_2}\right)\|Z, \xi_1, \xi_2\rangle \\
|Z\rangle_4 &= -\frac{2}{N(N-1)}\frac{\partial^2}{\partial\xi_2\partial\xi_1}\|Z, \xi_1, \xi_2\rangle.
\end{aligned} \tag{34}$$

By making use of equations (32) and (34), the differential realizations $D(G)$ of the $SPL(2,1)$ generators are constructed explicitly as follows:

$$\begin{aligned}
&D(Q_3) = -\frac{N}{2} + \frac{1}{2}\left(\xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2}\right) + Z\frac{\partial}{\partial Z} \qquad D(B) = \frac{1}{2}\left(\xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2}\right) \\
&D(Q_+) = \frac{\partial}{\partial Z} \qquad D(Q_-) = NZ - Z\xi_1\frac{\partial}{\partial\xi_1} - Z\xi_2\frac{\partial}{\partial\xi_2} - Z^2\frac{\partial}{\partial Z} \\
&D(V_+) = \frac{\partial}{\partial\xi_1} + \frac{1}{2}\xi_2\frac{\partial}{\partial Z} \qquad D(V_-) = \frac{1}{2}N\xi_2 - Z\frac{\partial}{\partial\xi_1} + \frac{1}{2}\xi_1\xi_2\frac{\partial}{\partial\xi_1} - \frac{1}{2}\xi_2 Z\frac{\partial}{\partial Z} \\
&D(W_+) = \frac{\partial}{\partial\xi_2} + \frac{1}{2}\xi_1\frac{\partial}{\partial Z} \qquad D(W_-) = \frac{1}{2}N\xi_1 - Z\frac{\partial}{\partial\xi_2} - \frac{1}{2}\xi_1\xi_2\frac{\partial}{\partial\xi_2} - \frac{1}{2}\xi_1 Z\frac{\partial}{\partial Z}.
\end{aligned} \tag{35}$$

For example, for $D(W_-)$, we have

$$W_-\|Z, \xi_1, \xi_2\rangle = W_-\left[|Z\rangle_1 + \frac{1}{\sqrt{2}}N\xi_1|Z\rangle_2 + \frac{1}{\sqrt{2}}N\xi_2|Z\rangle_3 - \frac{1}{2}N(N-1)\xi_1\xi_2|Z\rangle_4\right]$$

$$
\begin{aligned}
&= -\frac{1}{\sqrt{2}} NZ|Z\rangle_3 + \frac{1}{\sqrt{2}} N\xi_1 \frac{1}{\sqrt{2}} \frac{1}{N}\left(N - Z\frac{\partial}{\partial Z}\right)|Z\rangle_1 \\
&\quad - \frac{1}{\sqrt{2}} N\xi_1 \frac{1}{\sqrt{2}}(N-1)Z|Z\rangle_4 - \frac{1}{2}N(N-1)\xi_1\xi_2 \\
&\quad \times\left(-\frac{1}{\sqrt{2}}\frac{1}{N-1}\left(N-1-Z\frac{\partial}{\partial Z}\right)\right)|Z\rangle_3 \\
&= \left(-\frac{1}{\sqrt{2}}NZ + \frac{1}{2\sqrt{2}}N\xi_1\xi_2\left(N-1-Z\frac{\partial}{\partial Z}\right)\right) \\
&\quad \times\frac{\sqrt{2}}{N}\left(\frac{\partial}{\partial\xi_2} - \xi_1\frac{\partial}{\partial\xi_1\partial\xi_2}\right)\|Z,\xi_1,\xi_2\rangle \\
&\quad + \frac{1}{2}\xi_1\left(N - Z\frac{\partial}{\partial Z}\right)\left(1 - \xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2} + \xi_1\xi_2\frac{\partial}{\partial\xi_2\partial\xi_1}\right)\|Z,\xi_1,\xi_2\rangle \\
&\quad + \xi_1 Z\frac{\partial}{\partial\xi_2\partial\xi_1}\|Z,\xi_1,\xi_2\rangle \\
&= \left(\frac{1}{2}N\xi_1 - Z\frac{\partial}{\partial\xi_2} - \frac{1}{2}\xi_1 Z\frac{\partial}{\partial Z} - \frac{1}{2}\xi_1\xi_2\frac{\partial}{\partial\xi_2}\right)\|Z,\xi_1,\xi_2\rangle. \qquad (36)
\end{aligned}
$$

It is clear that the aforementioned realizations are inhomogeneous. Therefore, they may be of use for quasi-exactly solvable problems in quantum mechanics.

We have constructed the supercoherent state of the $SPL(2,1)$ superalgebra. We also have calculated the matrix elements of the $SPL(2,1)$ generators. The new inhomogeneous differential realizations of the $SPL(2,1)$ generators have been obtained in the supercoherent state space.

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